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A less restrictive Briançon–Skoda theorem with coefficients

Ian M. Aberbach, Aline Hosry*

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

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ABSTRACT

The Briançon–Skoda theorem in its many versions has been studied by algebraists for several decades. In this paper, under some assumptions on an F-rational local ring (R, \mathfrak{m}) , and an ideal I of R of analytic spread ℓ and height $g < \ell$, we improve on two theorems by Aberbach and Huneke. Let J be a reduction of I . We first give results on when the integral closure of $I^{\ell+w}$ is contained in the product $J^{w+1}I_{[\ell-1]}$, for any integer $w \geq 0$, where, given any primary decomposition of I , $I_{[\ell-1]}$ is the intersection of the primary components of I of height at most $\ell - 1$. In the case that R is also Gorenstein, we give results on when the integral closure of $I^{\ell-1}$ is contained in J .

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1. Introduction

In this paper, all rings are assumed to be commutative and Noetherian with identity.

The theorem of Briançon and Skoda was first proved in an analytic setting. Namely, let $O_n = \mathbb{C}\{z_1, \dots, z_n\}$ be the ring of convergent power series in n variables over the field of complex numbers. Let $f \in O_n$ be a non-unit (i.e., f vanishes at the origin), and let $J(f) = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})O_n$ be the Jacobian ideal of f . Then one can see that $f \in \overline{J(f)}$, the integral closure of $J(f)$ (cf., e.g., [10], Corollary 7.1.4). Since the integral closure of any ideal is always contained in the radical of that same ideal, in particular there is an integer k such that $f^k \in J(f)$. John Mather raised the following question: Does there exist an integer k that works for all non-units f ?

Briançon and Skoda first answered this question affirmatively by proving that the n th power of f lies in $J(f)$. This is an immediate result of the following theorem, proved by analytic methods:

* Corresponding author.

E-mail addresses: aberbachi@missouri.edu (I.M. Aberbach), aline.hosry@mail.missouri.edu (A. Hosry).

Theorem 1.1. (See [7].) Let $I \subseteq O_n$ be an ideal generated by ℓ elements. Then for all $w \geq 0$,

$$\overline{I^{\ell+w}} \subseteq I^{w+1}.$$

Since $f \in \overline{J(f)}$, $f^n \in \overline{J(f)}^n \subseteq \overline{J(f)^n} \subseteq J(f)$, by applying the Briançon and Skoda theorem for $I = J(f)$, which has at most n generators (taking w to be zero). Hence $f^n \in J(f)$ and Mather's question is answered.

Lipman and Sathaye proved that this purely algebraic result can be extended to arbitrary regular local rings as follows:

Theorem 1.2. (See [12].) Let (R, \mathfrak{m}) be a regular local ring and suppose that I is an ideal of R generated by ℓ elements. Then for all $w \geq 0$,

$$\overline{I^{\ell+w}} \subseteq I^{w+1}.$$

Lipman and Teissier were partially able to extend this theorem to pseudo-rational rings [13], while Aberbach and Huneke were able to prove the theorem for F -rational rings and rings of F -rational type in the equicharacteristic case [3].

Initially motivated by trying to understand the relationship between the Cohen–Macaulayness of the Rees ring of I , $R[It]$, and the associated graded ring of I , $\text{gr}_I R$, various authors (see, e.g., [3–6, 8, 11, 15]) have studied the coefficients involved in the Briançon–Skoda theorem. More specifically, if $J = (a_1, \dots, a_\ell)$ is a minimal reduction of I and $z \in \overline{I^\ell} = \overline{J^\ell}$, then when we write $z = \sum_{i=1}^{\ell} r_i a_i$, we may ask where the coefficients r_i lie. For instance, can we say that they lie in I ? Heuristically, when $\text{ht}(I) < \ell$, there is reason to have this occur. One such result is Theorem 3.6 of [4]. We are able to substantially reduce the necessary hypotheses needed in that paper. Explicitly, we prove the following result (see the next section for the definition of $I_{[\ell-1]}$):

Theorem 3.8. Let (R, \mathfrak{m}) be an F -rational, Cohen–Macaulay local ring (e.g., an excellent F -rational local ring), $I \subseteq R$ an ideal of analytic spread ℓ and positive height $g < \ell$, and let J be any reduction of I . Then for any integer $w \geq 0$,

$$\overline{I^{\ell+w}} \subseteq J^{w+1} I_{[\ell-1]}.$$

We are also interested in reducing the power ℓ of I in $\overline{I^\ell} \subseteq J$. This is possible in Gorenstein rings, with the aid of local duality, using ideas first utilized in [9]. More precisely, we improve on Theorem 4.1 of [3], due to Aberbach and Huneke, to show:

Theorem 4.2. Let (R, \mathfrak{m}) be an F -rational Gorenstein local ring of dimension d and characteristic $p > 0$. Suppose that I is an ideal of height g and analytic spread $\ell > g$. Assume that $I = I_{[\ell-1]}$ and that R/I has depth at least $d - \ell + 1$. Then for any reduction J of I , we have $\overline{I^{\ell-1}} \subseteq J$.

2. Preliminary results

In this section, we review some of the definitions and results that will be used in this paper.

Let (R, \mathfrak{m}) be a Noetherian local ring and let I be an ideal of R . An ideal $J \subseteq I$ is a *reduction* of I if there exists an integer n such that $JI^n = I^{n+1}$ [14]. The least such integer is the *reduction number* of I with respect to J . A reduction J of I is called a *minimal reduction* if J is minimal with respect to inclusion among reductions. When (R, \mathfrak{m}) is local with infinite residue field, every minimal reduction J of I has the same number of minimal generators. This number is called the *analytic spread* of I , denoted by $\ell(I)$, and we always have the inequalities $\text{ht}(I) \leq \ell(I) \leq \dim R$. The *analytic deviation* of I , denoted by $ad(I)$, is the difference between the analytic spread of I and the height of I , i.e. $ad(I) = \ell(I) - \text{ht}(I)$. We also define I^{un} to be the intersection of the minimal primary components

of the ideal I (under this definition I^{un} has no embedded components but may have components of different heights or dimensions).

An element x of R is said to be in the *integral closure* of I , denoted by \bar{I} , if x satisfies an equation of the form $x^k + a_1 x^{k-1} + \cdots + a_k = 0$, where $a_i \in I^i$ for $1 \leq i \leq k$. If an ideal $J \subseteq I$ is a reduction, then $\bar{J} = \bar{I}$.

Let R be a Noetherian ring of prime characteristic $p > 0$ and let q be a varying power of p . Denote by R^0 the complement of the union of the minimal primes of R and let I be an ideal of R . Define $I^{[q]} = (i^q \mid i \in I)$, the ideal generated by the q th powers of all the elements of I . The *tight closure* of I is the ideal $I^* = \{x \in R \mid \text{for some } c \in R^0, cx^q \in I^{[q]}, \text{ for } q \gg 0\}$. We always have that $I \subseteq I^* \subseteq \bar{I}$. If $I^* = I$, then the ideal I is said to be *tightly closed*. We say that elements x_1, \dots, x_n of R are *parameters* if the height of the ideal generated by them is at least n (we allow them to generate the whole ring, in which case the height is said to be ∞). The ring R is said to be *F-rational* if the ideals generated by parameters are tightly closed.

The remaining ingredients of this section are from [3] and [4].

Definition 2.1. (See [4], Definition 2.10.) Let R be a Noetherian local ring and let I be an ideal of height g . We say that a reduction $J = (a_1, \dots, a_\ell)$ of I is generated by a *basic generating set* if for all prime ideals P containing I such that $i = \text{ht}(P) \leq \ell$, $(a_1, \dots, a_i)_P$ is a reduction of I_P .

When the residue field of R is infinite, there always exist such basic generating sets, and furthermore, $\text{ht}((a_1, \dots, a_i)I^n : I^{n+1} + I) \geq i + 1$ for $n \gg 0$ (cf., [6], Lemma 7.2).

The next proposition plays a crucial role for the entire subsequent discussion.

Proposition 2.2. (See [3], Proposition 3.2.) Let (R, \mathfrak{m}) be an equidimensional and catenary local ring with infinite residue field and let $I \subseteq R$ be an ideal of height g and analytic spread ℓ . Let $J \subseteq I$ be a minimal reduction of I . We assume that $J = (a'_1, \dots, a'_\ell)$ is generated by a basic generating set as in Definition 2.1 above. Let N and w be fixed integers, and suppose that for $g + 1 \leq i \leq \ell$, we are given finite sets of primes $\Lambda_i = \{Q_{ji}\}$ all containing I and of height i . Then there exist elements a_1, \dots, a_ℓ and t_{g+1}, \dots, t_ℓ such that the following conditions hold. (We set $t_i = 0$ for $i \leq g$ for convenience.)

1. $a_i \equiv a'_i$ modulo I^2 .
2. For $g + 1 \leq i \leq \ell$, $t_i \in \mathfrak{m}^N$.
3. $b_1, \dots, b_g, b_{g+1}, \dots, b_\ell$ are parameters, where $b_i = a_i + t_i$.
4. The images of t_{g+1}, \dots, t_ℓ in R/I are part of a system of parameters.
5. There is an integer M such that $t_{i+1} \in (J_i^n I^M : I^{M+n})$ for all $0 \leq n \leq w + \ell$ where $J_i = (a_1, \dots, a_i)$.
6. $t_{i+1} \notin \bigcup_j Q_{ji}$, the union being over the primes in Λ_i .

Remark 2.3. We have altered the statement made in part 4 of Proposition 2.2 from that of the original, but the given statement holds.

Proposition 2.2 allows us to choose a parameter ideal, \mathfrak{m} -adically as close to I as desired, for which a sort of Briançon–Skoda result applies. Specifically:

Theorem 2.4. (See [3], Theorem 3.3.) Let (R, \mathfrak{m}) be an equidimensional and catenary local ring of characteristic p having an infinite residue field. Let I be an ideal of analytic spread ℓ and positive height g . Let J be a minimal reduction of I . Fix w and $N \geq 0$. Choose a_i and t_i as in Proposition 2.2. Set $\mathfrak{A} = B_\ell = (b_1, \dots, b_g, \dots, b_\ell)$. Then

$$\overline{I^{\ell+w}} \subseteq (\mathfrak{A}^{w+1})^*.$$

One of our main goals in this paper is to generalize the next theorem, which is due to Aberbach and Huneke. We recall that an ideal I satisfies a property generically if I_P satisfies that same property for every minimal prime P of I .

Theorem 2.5. (See [4], Theorem 3.6.) Let (R, \mathfrak{m}) be an excellent F -rational local ring, $I \subseteq R$ an ideal with reduction J . Let $g = \text{ht}(I) < \ell = \ell(I)$. Suppose that

- R/I is equidimensional,
- R/I^{un} satisfies $S_{\ell-g-1}$, and
- I is generically of reduction number at most one.

Then $\overline{I}^{\ell} \subseteq JI^{\text{un}}$.

In particular if R/I is equidimensional, I generically has reduction number at most one and I has analytic deviation 2, then $\overline{I}^{\ell} \subseteq JI^{\text{un}}$.

3. The main theorem

We will show that the two hypotheses in Theorem 2.5, Serre's condition $S_{\ell-g-1}$ on R/I^{un} , and the assumption that I is an ideal generically of reduction number at most one, are not necessary. We will also show that I^{un} may be replaced by a (potentially) smaller ideal, which in many instances, also allows us to remove the hypothesis that I is unmixed.

To set the stage, we first dispense with a number of preliminary results.

Lemma 3.1. Let R be a Noetherian ring, J an ideal of R , and x an element of R . Then there exists a positive integer M such that $J :_R x^{\infty} = J :_R x^M$.

Proof. $J :_R x \subseteq J :_R x^2 \subseteq \dots$ is an increasing sequence of ideals in the Noetherian ring R . \square

In the proofs below, we will often rename x^M back to x , and assume that $J :_R x^{\infty} = J :_R x$.

For the objectives of the main result, the succeeding proposition is pivotal. It is, in some sense, the reason that we can drop some of the earlier hypotheses to extend previous theorems of Briançon-Skoda type.

Proposition 3.2. Let R be a Noetherian ring and I an ideal of R . Let t_1 be an element of R such that its image is regular in R/I . Then for any elements t_2, \dots, t_n of R , there exist elements u_2, \dots, u_n of R such that for all $2 \leq i \leq n$, u_i is a power of t_i , and

$$(I, u_2^2, \dots, u_n^2) : t_1 \subseteq (I, u_2, \dots, u_n).$$

Proof. Let s_1 be the image of t_1 in $S = R/I$ and set $S_1 = S/(s_1)$.

By Lemma 3.1, we can pick $u_2 \in R$, a power of t_2 , such that its image s_2 in S_1 satisfies $0 :_{S_1} s_2^{\infty} = 0 :_{S_1} s_2$. For $3 \leq i \leq n$, pick $u_i \in R$, a power of t_i , such that its image s_i in S_1 satisfies

$$(s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^{\infty} = (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i.$$

For convenience, we are using the same notation for the elements s_i in the rings S and S_1 .

We claim that for $2 \leq i \leq n$, if $w \in (s_2^2, \dots, s_i^2) :_S s_1$, there exists v_i such that

$$w - v_i s_i \in (s_2^2, \dots, s_{i-1}^2) :_S s_1.$$

To prove the claim, let w be in $(s_2^2, \dots, s_i^2) :_S s_1$, then one can write that

$$s_1 w = \alpha_2 s_2^2 + \dots + \alpha_i s_i^2. \quad (3.1)$$

Thus $\alpha_2 s_2^2 + \cdots + \alpha_i s_i^2 = 0$ in S_1 , which implies that

$$\alpha_i \in (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^2 \subseteq (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^\infty = (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i.$$

Hence $\alpha_i \in (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i$, or equivalently $\alpha_i s_i \in (s_1, s_2^2, \dots, s_{i-1}^2) S$.

Write $\alpha_i s_i = v_i s_1 + x_i$ where $x_i \in (s_2^2, \dots, s_{i-1}^2)$. We get $\alpha_i s_i^2 = v_i s_1 s_i + x_i s_i$. Replacing this expression for $\alpha_i s_i^2$ back into (3.1), and combining the terms involving s_1 , we see that $s_1(w - v_i s_i) \in (s_2^2, \dots, s_{i-1}^2)$, which implies that $w - v_i s_i \in (s_2^2, \dots, s_{i-1}^2) :_S s_1$, as desired.

Therefore, we can conclude that if $w \in (s_2^2, \dots, s_n^2) :_S s_1$, there exist v_2, \dots, v_n such that $s_1(w - v_n s_n - \cdots - v_2 s_2) = 0$ in S .

But s_1 is a nonzerodivisor in S , so $w - v_n s_n - \cdots - v_2 s_2 = 0$ in S , implying that $w \in (s_2, \dots, s_n) S$. Therefore $(s_2^2, \dots, s_n^2) :_S s_1 \subseteq (s_2, \dots, s_n) S$, or equivalently, $(I, u_2^2, \dots, u_n^2) :_R t_1 \subseteq (I, u_2, \dots, u_n) R$ as desired. \square

Definition 3.3. If I has height g , then given $k \geq g$, set $S_k = R \setminus \bigcup \{P \mid P \in \text{Ass}_R(R/I), \text{ht}(P) \leq k\}$. We define $I_{[k]} = IS_k^{-1} R \cap R$. This means that given a primary decomposition of I , I_k is the intersection of the primary components of I of height at most k .

With the above terminology, the ambition of this section is to obtain a coefficient theorem of the form $\overline{I^{\ell+w}} \subseteq J^{w+1} I_{[k]}$, for some k depending on the ideal I .

Let (R, \mathfrak{m}) be an equidimensional and catenary local ring with infinite residue field and let $I \subseteq R$ be an ideal of height g and analytic spread ℓ . Let $J \subseteq I$ be a minimal reduction of I . We assume that $J = (a'_1, \dots, a'_\ell)$ is generated by a basic generating set as in Definition 2.1. Let N and w be fixed integers, and suppose that for $g+1 \leq i \leq \ell$ we are given finite sets of primes $\Lambda_i = \{Q_{ji}\}$ all containing I and of height i . By combining the previous results, we acquire the following:

Proposition 3.4. *With the above assumptions, there exist elements a_1, \dots, a_ℓ generating J and t_{g+1}, \dots, t_ℓ of R such that conditions 1 through 6 of Proposition 2.2 hold.*

In addition, we can choose the elements t_{g+1}, \dots, t_ℓ in R such that

$$(I_{[\ell-1]}, t_{g+1}^2, \dots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{[\ell-1]}, t_{g+1}, \dots, t_{\ell-1}).$$

Proof. Pick elements a_1, \dots, a_ℓ and t_{g+1}, \dots, t_ℓ in R as in Proposition 2.2. If necessary, replace $t_{g+1}, \dots, t_{\ell-1}$ by higher powers so that

$$(I_{[\ell-1]}, t_{g+1}^2, \dots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{[\ell-1]}, t_{g+1}, \dots, t_{\ell-1}).$$

This is possible since on the one hand properties 1–6 of Proposition 2.2 remain true after replacing $t_{g+1}, \dots, t_{\ell-1}$ by higher powers. And on the other hand, we can apply Proposition 3.2 once we check that t_ℓ is a nonzerodivisor in $R/I_{[\ell-1]}$. But this is true by property 6 of Proposition 2.2 if we set $\Lambda_{\ell-1}$ to be any finite set of height $\ell-1$ primes whose union contains all associated primes of I with height at most $\ell-1$. \square

A few more results are needed before we can give a proof of our main theorem in this section.

Let (R, \mathfrak{m}) be an equidimensional and catenary local ring of characteristic p , having an infinite residue field. Let I be an ideal of analytic spread ℓ and positive height g . Let J be a minimal reduction of I . Fix integers w and $N \geq 0$, and choose a_1, \dots, a_ℓ and t_{g+1}, \dots, t_ℓ as in Proposition 2.2. For $i = 1, \dots, \ell$, set $b_i = a_i + t_i$ ($t_i = 0$ for $i \leq g$), $J_i = (a_1, \dots, a_i)$, and $B_i = (b_1, \dots, b_i)$.

Lemma 3.5. *With the above assumptions, there exists an element $c \in I^M \cap R^0$ (where M is the integer from condition 5 of Proposition 2.2) such that the following conditions hold:*

1. For any $g + 1 \leq j \leq \ell$ and $1 \leq k \leq w + \ell$, $ct_j^q I^{kq} \subseteq J_{j-1}^{kq}$, for any power q of p .
2. For all $g \leq i \leq \ell$ and $0 \leq r \leq w$, we have $c^{i-g} J_i^{(i+r)q} \subseteq (B_i^{r+1})^{[q]}$, for any power q of p .

Proof. This lemma combines useful facts that were presented in the proof of Theorem 2.4. For their proofs, refer to the proof of Theorem 3.3 in [3]. \square

The next result is a generalization of Lemma 4.3 in [3].

Lemma 3.6. Under the above assumptions, assume that $g < \ell$ and let $w \geq -1$ be any integer. Then for all $g + 1 \leq j \leq \ell$, we have

$$t_j \overline{I^{\ell+w}} \subseteq (B_{j-1}^{\ell+w-j+2})^*.$$

Proof. Fix j between $g + 1$ and ℓ and choose an integer $m \geq \ell + w - j + 1 \geq 0$. Let $z \in \overline{I^{\ell+w}}$. Then there exists an element $d \in R^0$ such that $dz^q \in I^{(\ell+w)q}$, for $q \gg 0$. Also, choose $c \in I^M \cap R^0$ satisfying the conclusions of Lemma 3.5.

Hence, $dc^{j-g} t_j^q z^q \in t_j^q c^{j-g} I^{(\ell+w)q} = c^{j-g-1} ct_j^q I^{(\ell+w)q}$. Apply Lemma 3.5(1) and (2) to obtain $dc^{j-g} t_j^q z^q \in c^{j-1-g} J_{j-1}^{(\ell+w)q} \subseteq (B_{j-1}^{\ell+w-j+2})^{[q]}$ when taking $g \leq i = j - 1 \leq \ell$ and $0 \leq r = \ell + w - j + 1 \leq m$ (here we apply (2) of Lemma 3.5 for the integer m chosen as above, instead of w).

Thus, $dc^{j-g} t_j^q z^q \in (B_{j-1}^{\ell+w-j+2})^{[q]}$ which implies that $t_j z \in (B_{j-1}^{\ell+w-j+2})^*$, since the element dc^{j-g} is in R^0 . Therefore, we conclude that $t_j \overline{I^{\ell+w}} \subseteq (B_{j-1}^{\ell+w-j+2})^*$. \square

Remark 3.7. In Theorem 2.4 and Lemmas 3.5 and 3.6, if one replaces any $b_i = a_i + t_i$ by $a_i + t_i^2$, the conclusions remain unchanged. This is true because by raising any t_i to a higher power, conditions 1 through 6 of Proposition 2.2 still hold.

The next theorem generalizes Aberbach and Huneke's Theorem 3.6 of [3], stated here as Theorem 2.5. It shows that Serre's condition, the assumption that R/I is equidimensional and the generic reduction number hypothesis are superfluous. In the proof that we present, we make the appropriate modifications to Aberbach and Huneke's proof of Theorem 2.5.

Theorem 3.8. Let (R, \mathfrak{m}) be an F -rational, Cohen–Macaulay local ring (e.g., an excellent F -rational local ring), $I \subseteq R$ an ideal of analytic spread ℓ and positive height $g < \ell$, and let J be any reduction of I . Then for any integer $w \geq 0$,

$$\overline{I^{\ell+w}} \subseteq J^{w+1} I_{[\ell-1]}.$$

Remark 3.9. We employ the standard reduction to infinite residue field. Let X be a variable over (R, \mathfrak{m}) , then the ring $R[X]_{\mathfrak{m}R[X]}$ is a faithfully flat extension of R , having an infinite residue field. Using Lemma 8.4.2 of [10], many properties remain true after reduction to the infinite residue field case. The Gorenstein property is preserved when applying this reduction, since both type and Cohen–Macaulayness are preserved. Although the fact that the extension of an F -rational ring is still F -rational may be justified by earlier results, a succinct reference showing that the extension is F -rational is Theorem 4.3 of [2]. Thus by possibly renaming R , we may, in many cases, assume that R has an infinite residue field.

Proof. By Remark 3.9, assume that R has an infinite residue field and $J = (a_1, \dots, a_\ell)$ is a minimal reduction generated by a basic generating set.

Fix an integer N and choose $t_{g+1}, \dots, t_\ell \in \mathfrak{m}^N$ as in Proposition 3.4. Hence, t_{g+1}, \dots, t_ℓ satisfy the conditions of Proposition 2.2 as well as the inclusion

$$(I_{[\ell-1]}, t_{g+1}^2, \dots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{[\ell-1]}, t_{g+1}, \dots, t_{\ell-1}).$$

Set $b_k = a_k + t_k^2$ (we set $t_k = 0$ for $k \leq g$).

By Theorem 2.4, $\overline{I^{\ell+w}} \subseteq ((b_1, \dots, b_\ell)^{w+1})^*$, and the latter is tightly closed using Theorem 1.1 of [1] and the fact that b_1, \dots, b_ℓ are parameters. Hence we obtain that $\overline{I^{\ell+w}} \subseteq (b_1, \dots, b_\ell)^{w+1}$ which is the ideal generated by all monomials of degree $w+1$ in b_1, \dots, b_ℓ ; say that β_1, \dots, β_s are those monomials. Given $z \in \overline{I^{\ell+w}}$, we may then write $z = r_1\beta_1 + \dots + r_s\beta_s$, where $r_i \in R$, for $1 \leq i \leq s$. We aim to show that $r_i \in I_{[\ell-1]} + \mathfrak{m}^N$, for all $i = 1, \dots, s$.

For $1 \leq i \leq s$,

$$\begin{aligned} t_\ell r_i \beta_i &\in t_\ell (\overline{I^{\ell+w}}, \beta_1, \dots, \widehat{\beta_i}, \dots, \beta_s) \\ &\subseteq ((b_1, \dots, b_{\ell-1})^{w+2})^* + (\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_s), \quad \text{by Lemma 3.6} \\ &= (b_1, \dots, b_{\ell-1})^{w+2} + (\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_s). \end{aligned}$$

By combining the terms involving β_i , we conclude that

$$\begin{aligned} t_\ell r_i &\in (b_1, \dots, b_\ell) + (\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_s) : \beta_i \\ &\subseteq (b_1, \dots, b_\ell), \quad \text{since } b_1, \dots, b_\ell \text{ is a regular sequence} \\ &\subseteq (J, t_{g+1}^2, \dots, t_\ell^2) \\ &\subseteq (I_{[\ell-1]}, t_{g+1}^2, \dots, t_\ell^2). \end{aligned}$$

Now combine the terms containing t_ℓ to obtain that

$$\begin{aligned} r_i &\in (t_\ell) + (I_{[\ell-1]}, t_{g+1}^2, \dots, t_{\ell-1}^2) : t_\ell \\ &\subseteq (t_\ell) + (I_{[\ell-1]}, t_{g+1}, \dots, t_{\ell-1}), \quad \text{by Proposition 3.4.} \end{aligned}$$

Hence, $r_i \in (I_{[\ell-1]}, t_{g+1}, \dots, t_\ell) \subseteq I_{[\ell-1]} + \mathfrak{m}^N$, for all $i = 1, \dots, s$.

We conclude that $z \in I_{[\ell-1]} J^{w+1} + \mathfrak{m}^N$, as $r_i \in I_{[\ell-1]} + \mathfrak{m}^N$ and $\beta_i \in J^{w+1} + \mathfrak{m}^N$, for all $i = 1, \dots, s$.

As N was arbitrary, the Krull intersection theorem gives that $z \in J^{w+1} I_{[\ell-1]}$, finishing the proof of the theorem. \square

Remark 3.10. When R/I is equidimensional, every minimal prime of I has height $g < \ell$, so $I_{[\ell-1]}$ is the intersection of I^{un} with other primary components of I . Hence, $I_{[\ell-1]} \subseteq I^{un}$ in this case. Theorem 3.8, applied for $w = 0$, then implies that $\overline{I} \subseteq J I^{un}$. Therefore, Theorem 3.8 is a generalization of Aberbach and Huneke's Theorem 2.5, but removes the hypotheses involving Serre's condition on R/I^{un} , the generic reduction number of I , and the assumption that R/I is equidimensional.

Example 3.11. Let (R, \mathfrak{m}) be any F-rational local ring of characteristic p (e.g., a regular ring), and let $\{P_1, \dots, P_t\}$ be any set of distinct nonzero primes in R with no containment relations (i.e., if $i \neq j$ then $P_i \not\subseteq P_j$). Also, let n_1, \dots, n_t be any set of positive integers. We may assume, by renumbering, that $g = \text{ht}(P_1) \leq \text{ht}(P_i)$ for all i .

Consider the ideal $I = P_1^{(n_1)} \cap \dots \cap P_t^{(n_t)}$, which by assumption, has height g . Assume that $\ell(I) > g$, which is often the case even when I is equidimensional, and is always the case when some prime P_i has height greater than g (since $\ell(I) \geq \text{bight}(I) := \max\{\text{ht } P \mid P \text{ is minimal over } I\}$).

Applying Theorem 2.5 to I requires that the height of each P_i is also g . But it would also require significantly more, since the generic reduction numbers need to be at most one (if R is regular, for instance, this requires n_i to be at most 1 for each P_i of height four or greater and at most 2 otherwise). Even the above two requirements will not imply the Serre Condition $(S_{\ell-g-1})$, in general.

On the other hand, our Theorem 3.8 applies to every such ideal I .

4. A theorem for F-rational Gorenstein rings

In this section, we are interested in the cases where the power ℓ of I in the inclusion $\overline{I}^\ell \subseteq J$ (where J is a reduction of I), can be lowered. A cancellation theorem due to Huneke [9] inspired the main idea behind the proof of our next result. In particular, we extend another theorem of Aberbach and Huneke that states:

Theorem 4.1. (See [3], Theorem 4.1.) *Let (R, \mathfrak{m}) be an F-rational Gorenstein local ring of dimension d and having positive characteristic. Suppose that I is an ideal of height g and analytic spread $\ell > g$, with R/I Cohen–Macaulay. Then for any reduction J of I , $\overline{I}^{\ell-1} \subseteq J$.*

We extend Theorem 4.1 in the following way:

Theorem 4.2. *Let (R, \mathfrak{m}) be an F-rational Gorenstein local ring of dimension d and characteristic $p > 0$. Suppose that I is an ideal of height g and analytic spread $\ell > g$. Assume that $I = I_{[\ell-1]}$ and that R/I has depth at least $d - \ell + 1$. Then for any reduction J of I , we have $\overline{I}^{\ell-1} \subseteq J$.*

In particular, if $\ell = d$ and $I = I_{[\ell-1]}$, then $\overline{I}^{\ell-1} \subseteq J$.

Proof. The proof is a modification of the proof of Theorem 4.1 presented in [3].

Using Remark 3.9, we may assume that R has an infinite residue field and that J is a minimal reduction of I . Fix an integer $N \geq 0$, and set $\Lambda_{\ell-1}$ to be any finite set of primes of R of height $\ell - 1$ such that the union contains all associated primes of I of height at most $\ell - 1$. Choose a_1, \dots, a_ℓ and t_{g+1}, \dots, t_ℓ as in Proposition 2.2 (here we set $w = 0$).

For $1 \leq i \leq \ell$, let $b_i = a_i + t_i^2$ (with $t_i = 0$ for $i \leq g$), $J_i = (a_1, \dots, a_i)$ and $B_i = (b_1, \dots, b_i)$.

By our choice of $\Lambda_{\ell-1}$, t_ℓ is a nonzerodivisor in $R/I_{[\ell-1]} = R/I$. Since $\text{depth } R/I \geq d - \ell + 1$, we can pick elements $x_{\ell+1}, \dots, x_d$ in R such that $b_1, \dots, b_\ell, x_{\ell+1}, \dots, x_d$ is a regular sequence in R , and such that $t_\ell, x_{\ell+1}, \dots, x_d$ is a regular sequence in R/I .

By Proposition 3.2, we can replace $t_{g+1}, \dots, t_{\ell-1}$ by higher powers of themselves so that

$$(I, t_{g+1}^2, \dots, t_{\ell-1}^2, x_{\ell+1}, \dots, x_d) : t_\ell^2 \subseteq (I, t_{g+1}, \dots, t_{\ell-1}, x_{\ell+1}, \dots, x_d), \quad (4.1)$$

where to obtain this inclusion we use that t_ℓ^2 is regular modulo $(I, x_{\ell+1}, \dots, x_d)$.

Set $\mathfrak{A} = B_\ell + (x_{\ell+1}, \dots, x_d)$, $D = B_{\ell-1} : t_\ell^2$ and $K = B_{\ell-1} + (x_{\ell+1}, \dots, x_d)$. Note that $K : b_\ell = K$, since the elements involved form a regular sequence in R .

Let $Q = (I, t_{g+1}, \dots, t_{\ell-1}, x_{\ell+1}, \dots, x_d) + K : D$. We claim that $\mathfrak{A} : t_\ell^2 \subseteq Q$. Let $t_\ell^2 u \in \mathfrak{A}$ and write

$$t_\ell^2 u = w + vb_\ell = w + va_\ell + vt_\ell^2, \quad (4.2)$$

where $w \in K$. Then $t_\ell^2(u - v) \in B_{\ell-1} + (a_\ell, x_{\ell+1}, \dots, x_d)$, and hence

$$\begin{aligned} u - v &\in (B_{\ell-1} + (a_\ell, x_{\ell+1}, \dots, x_d)) : t_\ell^2 \subseteq (I, t_{g+1}^2, \dots, t_{\ell-1}^2, x_{\ell+1}, \dots, x_d) : t_\ell^2 \\ &\subseteq (I, t_{g+1}, \dots, t_{\ell-1}, x_{\ell+1}, \dots, x_d), \quad \text{by (4.1).} \end{aligned}$$

Thus, $u - v \in Q$. To show that $u \in Q$, it suffices to show that $v \in K : D \subseteq Q$. Let $d \in D$ and consider dv . By (4.2), $dt_\ell^2 u = dw + dvb_\ell$. But as $dt_\ell^2 \in B_{\ell-1}$, $dvb_\ell \in K$. Therefore, $dv \in K : b_\ell = K$, as

noted above. Consequently, $dv \in K$ and $v \in K : D \subseteq Q$. Hence, $u \in Q$ and this proves the claim that $\mathfrak{A} : t_\ell^2 \subseteq Q$. In particular it proves that $\mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_\ell^2)$.

Next, we show that $I^{\ell-1} \subseteq \mathfrak{A} : Q$. First, recall that $t_\ell^2 \overline{I^{\ell-1}} \subseteq B_{\ell-1}$, by Lemma 3.6. Thus, $\overline{I^{\ell-1}} \subseteq D$, and hence $\overline{I^{\ell-1}}(K : D) \subseteq D(K : D) \subseteq K \subseteq \mathfrak{A}$. Moreover, $II^{\ell-1} \subseteq \overline{I^\ell} \subseteq \mathfrak{A}^* = \mathfrak{A}$, by Theorem 2.4 and the fact that R is F-rational.

For $g+1 \leq j \leq \ell-1$, Lemma 3.6 implies that

$$t_j \overline{I^{\ell-1}} \subseteq ((B_{j-1}^2)^{\ell-j+1})^* \subseteq B_{j-1}^* = B_{j-1} \subseteq \mathfrak{A}.$$

Consequently, $(t_{g+1}, \dots, t_{\ell-1}) \overline{I^{\ell-1}} \subseteq \mathfrak{A}$. Therefore, we have proved that $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q$.

Finally, $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_\ell^2) = (\mathfrak{A} : t_\ell^2)$, by local duality. Hence,

$$\overline{I^{\ell-1}} \subseteq (J, t_{g+1}^2, \dots, t_{\ell-1}^2, t_\ell^2, x_{\ell+1}, \dots, x_d) \subseteq J + \mathfrak{m}^N.$$

An application of the Krull intersection theorem proves that $\overline{I^{\ell-1}} \subseteq J$. \square

Remark 4.3. If I is unmixed and equidimensional (e.g., R/I is CM) in an F-rational Gorenstein ring then $I = I_{[\ell-1]}$. Therefore, Theorem 4.2 is a generalization of Theorem 4.1.

Example 4.4. Let (R, \mathfrak{m}) be any F-rational Gorenstein local ring, and let $P \subseteq R$ be a prime ideal such that $\ell(P) = \dim R$, and R/P is a two-dimensional, non-CM ring. Then Theorem 4.2 applies to P , while Theorem 4.1 does not.

We close this section with a question for which a positive answer would allow us to enhance the conclusion in Theorem 4.2.

Question 4.5. Suppose that $\ell - g \geq 2$. Is it possible to improve on Theorem 4.2 to obtain that $\overline{I^{\ell-2}} \subseteq J$? If, in the proof of Theorem 4.2, we could show that, in fact, $\overline{I^{\ell-1}} \subseteq B_\ell + (x_{\ell+1}, \dots, x_d)$, then one could extend the result.

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